Classical Solution of Two Dimensional R^2-Gravity and Cross-Over Phenomenon *

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Abstract

Two dimensional quantum R^2-gravity and its phase structure are examined in the semiclassical approach and compared with the results of the numerical simulation. Three phases are succinctly characterized by the effective action. A classical solution of R^2-Liouville equation is obtained by use of the solution of the ordinary Liouville equation. The partition function is obtained analytically. A toatal derivative term (surface term) plays an important role there. It is shown that the classical solution can sufficiently account for the cross-over transition of the surface property seen in the numerical simulation.

1 Introduction

Importance of the semiclassical approach to the quantum gravity has long been known. (For a recent review, see [1].) It is true as well in the two dimensional (2d) quantum gravity. Liouville theory, which is equivalent to the 2d quantum gravity in the conformal gauge, has be treateded semiclassically [2, 3]. In this paper we study 2d quantum R^2-gravity in the similar manner. The motivations for studying this model can be said as follows. Firstly, the ordinary 2d gravity is essentially based on the lagrangian: $\mathcal{L} = \sqrt{g}(\frac{1}{2\gamma}R\frac{1}{\Delta}R + \frac{1}{G}R + \mu)$. Because Einstein term,$\int d^2x\sqrt{g}R$

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,is a topological quantity, the dynamical effect comes only from the induced part ,$R\frac{1}{\Delta}R$. The lowest derivative-order 'kinetic' term ,made purely of metric, is R^2. If higher-derivative terms have some meaning in 2d quantum gravity, this model is worthy of study as the simplest one. Secondly, the simulation data of R^2-gravity, with high statistics, has recently appeared. This theory is a good testing model of the quantum gravity that can be compared with the numerical experiment. We can examine how some important procedures, such as (infra-red and ultra-violet) regularization and renormalization, of the field theory work in the model.

R^2-gravity,for Lorentzian metric, was first quantumly treated by T.Yoneya[4], where Hamilton-Jacobi equation in the superspace approach is exactly solved. Its importance as an regularization (of the ultra-violet behaviour) was suggested in [5]. One of us (S.I.) has shown its renormalizability using the background-field method[6] and obtained some renormalization-group beta-functions. Kawai and Nakayama(KN)[7] have treated the system based on the conformal field theory. Their approach will be compared with the present one in sect.5.

Another interesting approach to the quantum gravity is the lattice simulation. Since the method of the dynamical triangulation was invented for the Euclidean quantum gravity [8, 9, 10], the non-perturbative aspect of the quantum gravity has been vigorously analysed these 10 years. By this approach, the effect of R^2-term was examined by [11] in the early stage of the development of the simulation. Recently a cross-over phenomenon of the surface from the fractal phase to the 'flat' phase was clearly observed [12, 13]. The computer simulation of quantum gravity has been now greatly improved. Especially data of 2d quantum gravity become so accurate that they can be closely compared with the analytical prediction. We examine the recent computer-simulation data of the 2d R^2-gravity and present its theoretical interpretation, especially focus our attention on the cross-over transition.

The semiclassical approach was intensively applied to the quantization around an extended object (soliton, kink,instanton ,etc.) [14]. The advantage of this approach is that the whole physical situation is simply viewed in an effective action. In this approach the central role is played by the classical solution. Non-perturbative effects are taken into account by incorporating the non-trivial classical vacuum (Liouville solution in the present case), while the fluctuation around the solution is treated perturbatively. In analyzing the 2d R^2-quantum gravity semi-classically, we must first find the appropriate classical solution.

We take the Euclidean action,

$$
\begin{gathered}
S_{tot} = S_{gra} + S_m \quad , \quad S_{gra}[g;G,\beta,\mu] = \int d^2x\sqrt{g}(\tfrac{1}{G}R - \beta R^2 - \mu) \quad , \\
S_m[g,\Phi;c_m] = -\int d^2x\sqrt{g}(\tfrac{1}{2}\textstyle\sum_{i=1}^{c_m}\partial_a\Phi_i\cdot g^{ab}\cdot\partial_b\Phi_i) \quad , \quad (\ a,b=1,2\ ) \quad ,
\end{gathered} \tag{1}
$$

under the fixed area condition $A = \int d^2x\sqrt{g}$. Here G is the gravitaional coupling constant, μ is the cosmological constant , β is the coupling strength for R^2-term and Φ is the c_m- components scalar matter fields.

2 Semiclassical Quantization

By taking the conformal-flat gauge, $g_{ab} = e^{\varphi}\ \delta_{ab}$, the action (1) gives us,after integrating out the matter fields and Faddeev-Popov ghost, the following partition function[15].

$$
\begin{aligned}
\int \frac{\mathcal{D}g\mathcal{D}\Phi}{V_{GC}}\{exp\frac{1}{\hbar}S_{tot}\}\ \delta(\int d^2x\sqrt{g}-A) = exp\frac{1}{\hbar}(\frac{8\pi(1-h)}{G}-\mu A)\times Z[A] \quad , \\
Z[A]\equiv \int \mathcal{D}\varphi\ e^{+\frac{1}{\hbar}S_0[\varphi]}\ \delta(\int d^2x\ e^{\varphi}-A) \quad , \qquad (2)
\end{aligned}
$$

$$
S_0[\varphi] = \int d^2x\ (\frac{1}{2\gamma}\varphi\partial^2\varphi - \beta\ e^{-\varphi}(\partial^2\varphi)^2 + \frac{\xi}{2\gamma}\partial_a(\varphi\partial_a\varphi)\ ) \quad , \quad \frac{1}{\gamma} = \frac{1}{48\pi}(26-c_m) \qquad (3)
$$

where the relations for Einstein term and the cosmological term: $\int d^2x\sqrt{g}R = 8\pi(1-h)$, $h =$ number of handles, $\int d^2x\sqrt{g} = A$,are used. [1] V_{GC} is the gauge volume due to the general coordinate invariance. ξ is a free parameter. The total derivative term generally appears when integrating out the anomaly equation $\delta S_{ind}[\varphi]/\delta\varphi = \frac{1}{\gamma}\partial^2\varphi$. This term turns out to be very important. [2] We consider the manifold of a fixed topology of the sphere ,$h=0$, and with the finite area A. Furthermore we consider the case $\gamma > 0$ ($c_m < 26$). [3] $\hbar$ is Planck constant. [4]

Let us describe the thermo-dynamical consideration which will be crucial in later discussions. The Laplace transform of (2) is written as

$$
\hat{Z}[\lambda] = \int_0^{\infty} Z[A]e^{-\lambda A/\hbar}\ dA\ = \int \mathcal{D}\varphi\ exp[\ +\frac{1}{\hbar}\{S_o[\varphi]-\lambda\int d^2x\ e^{\varphi}\}] \quad . \qquad (4)
$$

$Z[A]$ is the micro-canonical partition function with the area A , while $\hat{Z}[\lambda]$ is the grand-canonical partiton function with the chemical potential λ . In the grand-canonical case, the average area is controled by fixing λ through the relation,

$$
< A_{op} > = \frac{1}{\hat{Z}}\frac{d}{d(-\lambda/\hbar)}\hat{Z}[\lambda] \equiv < \int d^2x e^{\varphi} >_{\hat{Z}} \quad , \quad A_{op}\equiv\int d^2x\ e^{\varphi} \quad . \qquad (5)
$$

Conversely, the micro-canonical partiton function can be obtained from $\hat{Z}[\lambda]$ by the inverse Laplace transformation,

$$
Z[A] = \int\frac{d\lambda}{\hbar}\hat{Z}[\lambda]\ e^{+\lambda A/\hbar} \quad . \qquad (6)
$$

The integral should be carried out along an appropriate contour parallel to the imaginary axis. We write $\hat{Z}[\lambda]$ as

$$
\begin{aligned}
S_\lambda[\varphi] &\equiv S_0[\varphi] - \lambda\int d^2x\ e^{\varphi} \\
&= \int d^2x\ (\frac{1}{2\gamma}\varphi\partial^2\varphi - \beta\ e^{-\varphi}(\partial^2\varphi)^2\ + \frac{\xi}{2\gamma}\partial_a(\varphi\partial_a\varphi)\ - \lambda\ e^{\varphi}\ ) \quad , \qquad (7) \\
\hat{Z}[\lambda] &= \int\mathcal{D}\varphi\ exp\ \{\frac{1}{\hbar}S_\lambda[\varphi]\} \equiv\ exp\ \frac{1}{\hbar}\hat{\Gamma}[\lambda] \quad ,
\end{aligned}
$$

[1] The sign for the action is different from the usual convention as seen in (2).

[2] The uniqueness of this term, among all possible total derivatives, is shown in Discussions(sect.6).

[3] This is for the comparison with the 'classical limit' $c_m \to -\infty$. We can do the same analysis for $\gamma < 0$ without any difficulty.

[4] In this section only,we explicitly write $\hbar$ (Planck constant) in order to show the perturbation structure clearly.

where $\hat{\Gamma}(\lambda)$ is the effective action induced by $S_\lambda[\varphi]$. It can be calculated loop-wise by the semiclassical expansion : $\varphi(x) = \varphi_c(x;\lambda) + \sqrt{\hbar}\ \psi(x)$, with taking the solution of the classical field equation : $\frac{\delta}{\delta\varphi} S_\lambda[\varphi]\Big|_{\varphi_c} = 0$, as the background field. Then we have

$$\begin{aligned} \hat{Z}[\lambda] = &\ exp\ \tfrac{1}{\hbar} S_\lambda[\varphi_c] \times \textstyle\int \mathcal{D}\psi\ exp\{\tfrac{1}{2}\tfrac{\delta^2 S_\lambda}{\delta\varphi^2}|_{\varphi_c}\psi\psi + O(\sqrt{\hbar})\} \\ &\equiv exp\{\tfrac{1}{\hbar}\hat{\Gamma}^0[\lambda] + \hat{\Gamma}^1[\lambda] + O(\hbar)\} \quad , \\ \hat{\Gamma}[\lambda] = &\ \hat{\Gamma}^0[\lambda] + \hbar\hat{\Gamma}^1[\lambda] + O(\hbar^2) \quad , \quad \hat{\Gamma}^0[\lambda] \equiv S_\lambda[\varphi_c] \quad , \end{aligned} \tag{8}$$

where $\hat{\Gamma}^n[\lambda], (n \geq 1)$, is the quantum effects contributed from n-loop diagrams.

Writing the integrand of (6) as

$$\begin{aligned} Y[A,\lambda] &\equiv exp\ \frac{1}{\hbar}\Gamma^{eff}[A,\lambda] \equiv\ exp\ \frac{1}{\hbar}\{\ \hat{\Gamma}[\lambda] + \lambda A\ \} \\ &= \int \mathcal{D}\varphi\ exp\ \frac{1}{\hbar}[\ S_0[\varphi] - \lambda(\int d^2x e^{\varphi} - A)] \quad , \end{aligned} \tag{9}$$

the stationary point λ_c of $\Gamma^{eff}[A,\lambda]$ is determined by

$$\frac{d}{d\lambda}\Gamma^{eff}[A,\lambda]|_{\lambda_c} = \frac{d\hat{\Gamma}[\lambda_c]}{d\lambda_c} + A = 0 \quad , \quad \lambda_c = \lambda_c^0 + \hbar\lambda_c^1 + \cdots \quad . \tag{10}$$

It gives the dominant contribution to the contour integral of (6). This condition (10) coincides with the equation (5) if we identify A with $< A_{op} >$. It means the dominant contribution to the contour integral comes from the value of λ at which the grand partition function takes $< A_{op} >= A$. [5] Finally we obtain the approximate relations,

$$Z[A] \approx \frac{1}{\hbar}Y[A,\lambda_c] \quad , \quad Y[A,\lambda_c] =\ exp\ \frac{1}{\hbar}\Gamma^{eff}[A,\lambda_c] \approx exp\ \frac{1}{\hbar}\{\hat{\Gamma}^0[\lambda_c^0] + \lambda_c^0 A\} \quad , \tag{11}$$

where the former approximation is valid in the large system limit and the latter one is valid in the semi-classical limit. In the following, we will evaluate the leading part (order of $\hbar^0$) of $\Gamma^{eff}[A,\lambda_c]$: $\hat{\Gamma}^0[\lambda_c^0] + \lambda_c^0 A =\ S_{\lambda_c^0} + \lambda_c^0 A$.

3 Classical Configuration of $\mathbf{R}^2$-Gravity and Phase Structure

3.1 Classical Solution

The classical solution for $\beta = 0$ has been known as the Liouville solutions. (See ref.[3] for a recent review.) Furthermore, in the context of 2d quantum gravity or the string theory , it was studied by [16] and [17] . We consider here the general

[5] $\Gamma^{eff}[A,\lambda_c]$ is exactly the same as the ordinary (Schwinger's) effective action which is obtained by Legendre transformation of $\hat{Z}[\lambda]$ due to the change of the independent variable from λ to $A =< A_{op} >$.

case of β being an arbitrary real number. The classical equation $,\frac{\delta S_\lambda[\varphi]}{\delta\varphi}=0$, is explicitly written as

$$\frac{\delta S_\lambda[\varphi]}{\delta\varphi}=\frac{1}{\gamma}\partial^2\varphi+\beta\{e^{-\varphi}(\partial^2\varphi)^2-2\partial^2(e^{-\varphi}\partial^2\varphi)\}-\lambda e^{\varphi}=0 \quad . \tag{12}$$

We make the assumption of constant curvature for the solution. [6]

$$-\left.R\right|_{\varphi_c}=\ e^{-\varphi_c}\partial^2\varphi_c=\mathrm{const}\equiv\frac{-\alpha}{A} \quad , \tag{13}$$

where α is a dimensionless constant which should satisfy

$$\text{COND.1} \qquad \alpha^2\beta'-\frac{1}{\gamma}\alpha-\lambda A=0 \quad , \quad \beta'\equiv\frac{\beta}{A} \quad , \tag{14}$$

as the consequence of classical field equation (12). It has real solutions α when parameters β',λ and γ satisfy $D_1\equiv\frac{1}{\gamma^2}+4\beta'\lambda A\geq 0$. Since eq.(13) is the Liouville equation with the cosmological constant $-\frac{1}{\gamma}\frac{\alpha}{A}$ (which is negative for $\alpha>0$ and positive for $\alpha<0$ in the present case of $\gamma>0$), the present solution contains that of Refs.[16, 17] as the $\beta=0$ case.

In this paper we consider only the case of the positive curvature: $\alpha>0$. The spherically symmetric [7] solution of (13) is known to be (cf.[16, 17, 3]),

$$\varphi_c(r;\alpha)=-ln\ \{\frac{\alpha}{8}(1+\frac{r^2}{A})^2\} \quad , \quad r^2=(x^1)^2+(x^2)^2 \quad . \tag{15}$$

It gives $\int d^2x\sqrt{g}R\Big|_{\varphi_c}=-\int d^2x\ \partial^2\varphi_c=8\pi$, which says the manifold described by the solution (15) has the sphere topology. The area, $\int d^2x\sqrt{g}\Big|_{\varphi_c}=\int d^2x\ e^{\varphi_c}=\frac{8\pi}{\alpha}A$, can be interpreted as the effective area covered by the classical solution. The equations (14 -15) constitute a solution of (12).

$S_\lambda[\varphi_c]$ is given as

$$\begin{gathered}S_\lambda[\varphi_c]=(1+\xi)\tfrac{4\pi}{\gamma}\ ln\tfrac{\alpha}{8}-16\pi\alpha\beta'+C(A) \quad , \\ C(A)=\tfrac{8\pi(2+\xi)}{\gamma}+\tfrac{8\pi\xi}{\gamma}\{\ ln(1+L^2/A)-(L^2/A)/(1+(L^2/A))\ \} \quad ,\end{gathered} \tag{16}$$

where L is the infrared cut-off ($r^2\leq L^2$) introduced for the divergent volume intgral of the total derivative term. Note that $C(A)$ does not depend on β and α . For the analysis of the β-dependence of physical quantities, we may disregard $C(A)$. However ,for the A-dependence (such as that of $Z[A]$), $C(A)$ plays an important role. The eq. (10) at the classical level is written as,

$$\frac{dS_\lambda[\varphi_c]}{d\lambda}+A=\{\frac{4\pi}{\gamma}\frac{1}{\alpha}(1+\xi)-(16\pi\beta'+\frac{1}{\gamma})+2\beta'\alpha\}\frac{d\alpha}{d\lambda}=0 \quad , \tag{17}$$

[6] The importance of the constant-curvature solution will be commented on in Sect.6. Other solutions will not be considered. They correspond to different (classical) vacua from the present one.

[7] in the (x^1,x^2)-plane

where we have used a relation : $1 = \frac{d\lambda}{d\alpha}\frac{d\alpha}{d\lambda} = \frac{1}{A}(\ 2\alpha\beta' - \frac{1}{\gamma})\frac{d\alpha}{d\lambda}$, which is derived from (14). This equation fixes the stationary point which dominates in the contour integral (6);

$$\text{COND.2} \qquad 2\beta'\alpha^2 - (16\pi\beta' + \tfrac{1}{\gamma})\alpha + (1+\xi)\tfrac{4\pi}{\gamma} = 0 \quad , \tag{18}$$

which has two real solutions;

$$\alpha_c^{\pm} = \tfrac{1}{4\beta'}\{16\pi\beta' + \tfrac{1}{\gamma} \pm \sqrt{(16\pi\beta')^2 + \tfrac{1}{\gamma^2} - \xi\tfrac{32\pi}{\gamma}\beta'}\ \} \quad , \tag{19}$$

when the condition $D \equiv (16\pi\beta')^2 + \frac{1}{\gamma^2} - \xi\frac{32\pi\beta'}{\gamma} = (16\pi\beta' - \frac{\xi}{\gamma})^2 + \frac{1-\xi^2}{\gamma^2} \geq 0$ is satisfied. The relation (14) then determines $\lambda_c^{\pm}(\beta) \equiv \lambda(\beta, \alpha_c^{\pm}(\beta))$. Note that the determinant of the above quadratic equation is positive definite for all real β if we take ξ for the region : $-1 \leq\ \xi\ \leq\ +1$. We consider this case in the following.

In summary two unknown parameters α and λ are fixed by two conditions COND.1 and 2 ,and they are expreesed by three physical parameters β ,γ ,A and one free parameter ξ. In Fig.1 we plot $\alpha_c^{\pm}$,which is equal to the curvature$\times A$, as the function of $w \equiv 16\pi\beta'\gamma$. The solution of α_c^{+} is negative in the region of $\beta < 0$. This contradicts the present condition $\alpha > 0$. Furthermore the curvature and other physical quantities ,calculated using α_c^{+} , diverge as $\beta \to \pm 0$. These behaviours contradict the results of numerical simulation. Therefore we consider mainly α_c^{-}-solution in the following. (α_c^{+}-solution will be discussed in sect.5, in relation to KN's result.)

Fig.1 $A\times$ Curvature ,$\alpha_c^{\pm}$-branches, $w \equiv 16\pi\beta'\gamma,\ \xi = 0$

3.2 Analysis of α_c^{-}-Solution and Cross-Over Phenomenon

In Fig.2 the R^2-expectation value : $A < \int d^2x\sqrt{g}R^2 > = -\frac{\partial\Gamma^{eff}[A,\lambda_c]}{\partial\beta'}$ is shown in the Log-Log scale for $w > 0$. It clearly shows the transition similar to one observed in the numerical simulation. Later (in Fig.5) we will show the theoretical curve in the linear scale for all real w . This classical solution gives rather good agreement even in the negative w region. Fig.3 and Fig.4 show the string tension $\times\gamma A = \ \gamma\lambda_c A$,and the total free energy $\times\gamma\ = -\gamma\Gamma^{eff}[A, \lambda_c]$,respectively. [8] From the fact that the

[8] As for the figure of the total free energy (Fig.4), the β-independent part $C(A)$ is omitted.

effective area is given by the inverse of α_c^- (the effective area $\times\frac{1}{A} = \frac{1}{A}\int d^2x e^{\varphi_c} = 8\pi/\alpha_c^-$) and from the behaviour of α_c^- in Fig.1, we notice that the area covered by the classical configuration is not the same as as the area constrained by the δ-function in the micro- canonical partition function except the $\beta \rightarrow -\infty$ region. This has happened because we are approximating the fully-quantumly fluctuating manifold by a simple classical sphere whose configuration is specified only by the effective area $1/\alpha$ and the string tension λ . This characteristically shows the present effective action approach using $Y[A, \lambda]$ (9). This point will be discussed further in sect.5.

Fig.2 $A < \int d^2x\sqrt{g}R^2 > |_c$, Log-Log plot for $w > 0$,α_c^--branch, $\xi = 0$

Fig.3 $\gamma A\times$(String Tension) ,α_c^--branch, $\xi = 0$

Fig.4 $\gamma\times$(Total Free Energy) ,α_c^--branch, $\xi = 0$

The asymptotic behaviours of some physical quantities are listed in Table 1.

Phase	(C) $w \ll -1$	(B) $\lvert w\rvert \ll 1$	(A) $1 \ll w$
α_c^-	8π $+O(\lvert w\rvert^{-1})$	$4\pi(1+\xi)\{1$ $-\frac{1-\xi}{2}w\} + O(w^2)$	$\frac{4\pi(1+\xi)}{w}$ $+O(w^{-2})$
$-\frac{\partial\Gamma_-^{eff}}{\partial\beta'}$	$64\pi^2 + \frac{0}{\lvert w\rvert}$ $+O(w^{-2})$	$16\pi^2(1+\xi)\{3-\xi$ $-(1-\xi)^2 w\} + O(w^2)$	$\frac{64\pi^2(1+\xi)}{w}$ $+O(w^{-2})$
$\gamma\lambda_c^- A$	$-4\pi\lvert w\rvert\{1$ $+O(\frac{1}{\lvert w\rvert})\}$	$4\pi(1+\xi)\{-1$ $+\frac{3-\xi}{4}w\} + O(w^2)$	$-\frac{\pi}{w}(1+\xi)(3-\xi)$ $+O(w^{-2})$
$-\gamma\Gamma_-^{eff}$	$-4\pi\lvert w\rvert\{1$ $+O(\frac{1}{\lvert w\rvert})\} - \gamma C(A)$	$4\pi(1+\xi)\{1 - ln\ \frac{1+\xi}{2}$ $+\frac{3-\xi}{4}w\} + O(w^2) - \gamma C(A)$	$4\pi(1+\xi)\ ln\ w$ $+ \text{const} -\gamma C(A)$

Table 1 Asymp. behaviour of physical quantites for α_c^--solution.
$R > 0, w \equiv 16\pi\beta'\gamma, \gamma = \frac{48\pi}{26-c_m} > 0$ ($c_m < 26$). $C(A)$ is given by (16).

From these graphs and Table 1, we can observe three types of surfaces.

(A) Free Creased Surface; Large positive β ($w \gg 1$)
As β increases, the string tension decreases to zero (in the negative sign), $\gamma\lambda_c^- A \sim -\frac{\pi(1+\xi)(3-\xi)}{w}$, and dynamics is mainly governed by the 'kinetic' and total derivative terms (3). This phase is not influenced by the area condition or the 'potential' term $-\lambda e^{\varphi}$ in (9) or (7). The characteristic mass scale is $1/\sqrt{\beta}$ as shown in the asymptotic behaviour of Rieman curvature $R \propto \frac{1}{\beta}$ and of the string tension $\lambda_c^- \propto -\frac{1}{\beta}$. The asymptotic behaviour $Z[A] \sim A^{4\pi(1-\xi)/\gamma} \times e^{O(1/w)}$ shows the conformal behaviour . The surface is mildly 'creased' with the curvature of order $\frac{1}{\beta}$. As β increases, the size of the creases on the surface becomes large (the surface becomes less creased) and the 'effective area' increases. As β decreases, the surface becomes more creased and the 'effective area' decreases.
$\frac{1}{A}\int d^2x\ e^{\varphi_c} = \frac{8\pi}{\alpha} \sim 2w/(1+\xi)$ shows this situation. The data of the simulation well

fits with the above image. Firstly the predicted asymptotic behaviour $A < \int d^2x\sqrt{g}\ R^2 >\sim \frac{64\pi^2}{w}(1+\xi)$ well describe the data both qualitatively and quantitatively. (We will soon do the fitting with data in Sect.4.) Secondly the loop-length distribution[12] and the coordination number distribution[13] clearly shows the above image.

(B) Fractal surface; $\beta \approx 0$ ($|w| \ll 1$)
The string tension is finitely present and the sign is negative: $\gamma\lambda_c^- A \sim -4\pi(1+\xi) + 3\pi(1+\xi)(3-\xi)w$. The surface configuration is determined not only by the 'kinetic' and total-derivative terms but also by the 'potential' term. The two mass parameters (the coupling β and the area parameter A) are balanced in such a way that there is no charactersic mass-scale in this phase. All physical quantites behave linearly with respect to w. In particular the asymptotic behaviour: $A < \int d^2x\sqrt{g}\ R^2 >\sim 16\pi^2(1+\xi)\{3-\xi-(1-\xi)^2 w\}$ well describes the data of the computer simulation both qualitatively and quantitatively. The behaviour $Z[A]|_{w=0} \sim A^{-8\pi\xi/\gamma}$ shows the conformal one. The value of the curvature at $\beta = 0$ is $R \times A|_{w=0} = \alpha_c^-(w=0) = 4\pi(1+\xi)$. [9] The cross-over point between (B) and (A) is roughly obtained as the point where the approximation-condition for this region breaks down: $w_{C.O.} = 16\pi\gamma\beta'_{C.O.} \sim 1$. (We will soon define the point definitely and obtain the explicit expression .) Note that the cross-over point on β'-axis goes to $+\infty$ as $c_m \to -\infty$ (so-called 'classical' limit in 2d quantum gravity): $\beta'_{C.O} \sim \frac{1}{16\pi\gamma} = \frac{26-c_m}{16\times 48\pi^2} \to +\infty,\ c_m \to -\infty$.

(C) Strongly-Tensed Perfect Sphere; Large negative β ($w \ll -1$)
The constant value of the curvature $\alpha_c^- \sim 8\pi$,irrespective of the value ξ , implies this phase describes the 'perfect sphere'. [10] The asymptotic behaviours $\lambda \propto -\frac{|\beta|}{A^2}$, $R \propto \frac{1}{A}$ show the characteristic mass scales are $\frac{1}{\sqrt{A}}$ in addition to β. Dynamics is strongly influenced by the potential term. Both the string tension and the total free energy are negatively divergent as $\beta \to -\infty$. The surface is strongly tensed.

4 Role of Total Derivative Term ,Determination of ξ and Data Fit

Let us see more closely how much the present analytical prediction fits with the data and see the role of the total derivative term(ξ-term in (7)). All the graphs in sect.3 are evaluated at $\xi = 0$. The log-log plot of $-\frac{\partial\Gamma^{eff}[A,\lambda_c]}{\partial\beta'}$ (Fig.2) shows, at some point $w_c > 0$, the behaviour qualitatively changes from the linearly-descending line to the constant-line as we decrease w. We call the

[9] This value is compared with the expectation value obtained from the known exact coordination-number(q_i) distribution on lattice: $R_i a^2 = 2\pi\frac{6-q_i}{q_i}$, a^2 = unit area of a triangle, $< R_i a^2 >= 2\pi\sum_{q=3}^{\infty}\frac{6-q}{q}W(q) \approx 4\pi \times 0.117478$, $W(q) = 16\cdot(\frac{3}{16})^q\cdot\frac{(q-2)(2q-2)!}{q!(q-1)!}$. [11]

[10] This terminology 'perfect sphere' is used here in order to discrminate the configuration that the surface is, as its shape, a sphere from the configuration that the surface is topologically a sphere.

changing point ,w_c ,*cross-over point*. Let us define the point definitely and see its ξ-dependence. Those two straight lines are given as $-\frac{\partial\Gamma^{eff}[A,\lambda_c]}{\partial\beta'} \rightarrow$ $64\pi^2\frac{1+\xi}{w}$ as $w \rightarrow +\infty$, $-\frac{\partial\Gamma^{eff}[A,\lambda_c]}{\partial\beta'} \rightarrow 16\pi^2(1+\xi)(3-\xi)+O(w)$ as $w \rightarrow +0$. We can unambiguously define the crossing point of two asymptotic lines above as the cross-over point w_c , and get as $w_c(\xi) = \frac{4}{3-\xi}$. w_c moves in the range $1 \leq w_c \leq 2$ for the present choice of $\xi : -1 \leq \ \xi \ \leq \ +1$. This result shows the ξ-term determines the essential part of the theory.

For the $\beta = 0$ ($w = 0$) case, the partition function is exactly known as $Z_{exc}[A] = A^{\gamma_s - 3}, \gamma_s = (c_m - 1 - \sqrt{(1-c_m)(25-c_m)}\ )/12$.[5] The present approximate result should coincide with it at the 'classical' limit,$c_m \rightarrow -\infty$. This requirement gives $\xi = 1$.

Now we fit the present theoretical curve of $A < \int d^2x\sqrt{g}R^2 >$ with data of [12]. Three adjusting parameters (P_1, P_2, P_3) are necessary for the fit:

$$-\frac{\partial\Gamma^{eff}[A,\lambda_c]}{\partial\beta'} = \ P_1 \cdot Y \quad , \quad w = \ P_2 \cdot (X + P_3) \quad , \tag{20}$$

where $(w, -\frac{\partial\Gamma^{eff}[A,\lambda_c]}{\partial\beta'})$ is the theoretical scale (see Table 1) and (X, Y) is the scale of the simulation data. The meaning of the adjusting parameters are as follows: 1) P_1 adjusts the scale of the expectation value , $< \int d^2x\sqrt{g}R^2 >$, itself [11] ; 2) P_2 adjusts the scale of the 'width' of the phase (B) in the w-axis ; 3) P_3 adjusts the origin of the w-axis. [12] P_1 and P_2 should be positive, whereas P_3 may be positive,zero or negative. We can fix those parameters ,for each ξ ,by the use of three data points: $(X, Y) = (-100.0, 1.69265), (0.0, 0.70605), (100.0, 0.08781)$. In Fig.5, we plot the adjusted curves of $-\frac{\partial\Gamma^{eff}[A,\lambda_c]}{\partial\beta'}$,in the linear scale, for three typical values of ξ $(-0.99, 0.0, 0.99)$ with the simulation data. The parameters used in Fig.5 are listed in Table 2. We must realize that the the total derivative term greatly influences the final result.

Fig.5 Fit of $< \int d^2x\sqrt{g}R^2 >$ .Dots are data points.

[11] P_1 is the ambiguity of a multiplicative constant which appears in comparing an expectation value of a continuous theory with that of the corresponding lattice theory.

[12] This parameter P_3 reflects the renormalization (quantum) effect.

ξ	(1) -0.99	(2) 0.00	(3) 0.99
P_1	373.2	373.2	373.2
P_2	4.505×10^{-3}	0.1705	0.3365
P_3	-284.6	8.556	12.48

Table 2 Parameters used in the data fit of Fig.5

5 α_c^+-Solution and Kawai-Nakayama's Result

There exits a conformal approach to the present problem [7]. They treat the phase (A) in Table 1 as the conformal phase. Their result about the asymptotic behaviour of the partition function $Z[A]$ does not coincide with the present one. We discuss the origin of the discrepancy. The sharp contrast of the two approaches exists in the treatment of the area constraint: $\int d^2x\sqrt{g} = A$, and the topological constraint: $\int d^2x\sqrt{g}R = 8\pi$. (i) The present approach does not directly 'solve' the area costraint, whereas KN does it. (ii) We respect the topological constraint, whereas KN does not.

For (i), we introduce the parameter of the chemical potential λ ,which can be regarded as the 'Lagrangian multiplier' for the area-constraint as shown in (9) and is physically interpreted as the surface (or string) tension. The validity of this treatment in the semiclassical approach can be stated as follows. The 'effective' sphere (the classical solution (15)), which approximates the fully-quantum surface-configuration ,does not necessarily satisfy the area constraint $\int d^2x\sqrt{g} = A$. The constraint is satisfied only when the dominant configuration is near the perfect sphere which characteristically has the large surface -tension (positive or negative) and the characteristic mass scale of $\frac{1}{\sqrt{A}}$. [13] When the surface-tension is not large, the configuration is far from the perfect sphere and we cannot use the area-constraint on the leading configuration. In other words, the area-cnstraint must also be treated 'perturbatively' as far as the semiclassical approximation works correctly. For (ii), we have introduced the parameter α in (13) to give the variableness for the value of the constant curvature. This variableness gives, through the solution (15), the correct constraint for the topological quantity: $\int d^2x\sqrt{g}R|_{\varphi_c} = R|_{\varphi_c}\cdot\int d^2x\sqrt{g}|_{\varphi_c} = \frac{\alpha}{A}\times\frac{A}{\alpha}8\pi = 8\pi$.

In the analysis of previous sections,we have considered only α_c^-- solution which does not satisfy the area constraint for $w \gg 1$ (A-phase) : $\int d^2x\sqrt{g}|_{\varphi_c} \approx A\times\frac{2w}{1+\xi}$. As for α_c^+- solution, the following asymptotic behaviours are obtained for $w \gg 1$.

$$\alpha_c^+ = 8\pi + O(w^{-1}) \ , \ \int d^2x e^{\varphi_c^+} = A(1+O(w^{-1})) \ , \ \gamma\lambda_c^+ A = 4\pi w(1+O(w^{-1})), \quad \text{for} \quad w \gg 1 \quad . \tag{21}$$

This result shows, α_c^+-solution satisfies the area constraint for $w \gg 1$. We explain below that this phase describes the perfect sphere. As expected, we find exactly

[13] Phase (C) in sect.3 is the case. In the phase (B), the surface-configuration is *not* near the perfect sphere. In this phase, however, the area constraint is satisfied by virtue of the 'topological effect' due to ξ-term : $\int d^2x\sqrt{g}|_{\varphi_c} \approx \frac{2A}{1+\xi} = A$ for $\xi = 1$.

KN's result in this region. [14]

$$Z[A] \approx A^{-\frac{8\pi\xi}{\gamma}} exp\{-\frac{4\pi w}{\gamma}(1+O(w^{-1}))\} \quad , \quad \xi = 1 \quad . \tag{22}$$

We make remarks about other properties of the branch α_c^+. The asymptotic behaviours are listed in Table 3.

Phase	$w \ll -1$	$-1 \ll w < 0$	(E) $0 < w \ll 1$	(D) $1 \ll w$
α_c^+	<0 , not allowed	<0 , not allowed	$\frac{4\pi}{w}\{2+w(1-\xi)$ $+O(w^2)\}$	8π $+O(w^{-1})$
$-\frac{\partial\Gamma_+^{eff}}{\partial\beta'}$	/	/	$-\frac{64\pi^2}{w^2}\{1-(1+\xi)w$ $+O(w^2)\}$	$64\pi^2\{1+\frac{0}{w}$ $+O(w^{-2})\}$
$\gamma\lambda_c^+ A$	/	/	$\frac{4\pi}{w}\{-1+0\cdot w$ $+O(w^2)\}$	$4\pi w\{1$ $+O(w^{-1})\}$
$-\gamma\Gamma_+^{eff}$	/	/	$\frac{4\pi}{w}\{1+2w$ $+(1+\xi)w\ ln\ w$ $+O(w^2)\}-\gamma C(A)$	$4\pi w\{1$ $+O(w^{-1}\}$ $-\gamma C(A)$

Table 3 Asymp. behaviour of physical quantities for α_c^+-solution. $R>0, w \equiv 16\pi\beta'\gamma, \gamma = \frac{48\pi}{26-c_m} > 0$ $(c_m < 26)$. $C(A)$ is given by (16).

Each phase in Table 3 is explained as follows.

(D) Explosive Perfect Sphere; Large positive β $(w \gg 1)$
This phase describes the configuration of the strongly-expanding perfect sphere. The asymptotic behaviour of the string tension: $\gamma\lambda_c^+ A \sim 4\pi x \rightarrow +\infty (x \rightarrow +\infty)$, shows the surface is strongly forced expansively. [15] The asymptotic behaviour of the partition function is given above ,(22). The constant value of the curvature ($\alpha_c^+ \sim 8\pi$) corresponds to the perfect sphere with the radius$\sim \sqrt{A}$. The characteristic length scale is fixed by the area parameter A ,not by β. The total free energy is positively divergent ($-\gamma\Gamma_+^{eff} \sim 4\pi w$) as β increases to $+\infty$, therefore this configuration is not preferable. The predicted result,
$A < \int d^2x\sqrt{g}\ R^2 > \sim (64\pi^2)$ (constant), contradicts the data of the lattice simulation[12]. We conclude this phase does not describe the data.

(E) Degenerate Surface; Small positive β $(0 < w \ll 1)$
As β goes to $+0$, the curvature increases to $+\infty$ ($\alpha_c^+ \sim \frac{8\pi}{w}$) and the area decreases to $+0$ ($\frac{8\pi}{\alpha_c^+} \sim w$). This shows the surface is degenerate. [16] The radius of the 'effective' sphere is approximately $\sqrt{\beta}$. The characteristic length scale is controlled by $\sqrt{\beta}$, not by $\sqrt{A}$. The string tension becomes negatively divergent

[14] The relation between the present notation and the KN's is $32\pi\beta = 1/m^2$,where $1/m^2$ is the KN's notation for the higher derivative couplings.

[15] The phase (C) in sect.3 also describes the configuration of perfect sphere, but the string tension and the total free energy have the different sign.

[16] The behaviour of vanishing area makes us imagine that this phase describes,so-called, branched polymer.

$(\gamma\lambda_c^+ A \sim -\frac{4\pi}{w})$. It means the (degenerate) surface is strongly tensed. The partition function behaves asymptotically as $Z[A] \sim A^{-\frac{8\pi\xi}{\gamma}} exp(-\frac{4\pi}{\gamma}\frac{1}{w})$, which contradicts the known conformal result. The total free energy becomes positively divergent as $\beta \rightarrow +0$: $-\gamma\Gamma_+^{eff} \sim +\frac{4\pi}{w} \rightarrow +\infty$, therefore this phase is energetically unpreferable. The predicted behaviour of $A < \int d^2x\sqrt{g}\ R^2 > \sim -\frac{64\pi^2}{w^2}$ contradicts the simulation data . It is crucial that the solution is not connected with the $\beta < 0$ region, while the simulation data shows the physical quantities are continuously connected with the $\beta < 0$ region. This phase does not describe the simulation data.

As explained in (E), α_c^+-solution does have the bad behaviour for $w \rightarrow +0$,which cannot be accepted in the conformal approach. Similar bad behaviour is also noticed in KN's case.

α_c^+-solution shares similar properties with those of KN's. KN's solution looks to correspond to the α_c^+-solution in the present formalism, in particular for the $w \gg 1$ region.

6 Discussions

Some additional comments are in order.

1. We have pointed out the importance of the total derivative term in (3). There could be many other types of total derivative terms, but they are exculded as follows.

   (a) Higher-Derivative Terms: From the dimensional analysis, the higher-derivative terms vanish for the limit $L \rightarrow +\infty$,where L is the infrared-regularization parameter introduced in (16). For example: $\int d^2x\partial_a(\varphi_c\partial^2\partial_a\varphi_c) \sim\ (ln\ L)/L^2,\ \int d^2x\partial_a(\partial^2\varphi_c\cdot\partial_a\varphi_c) \sim\ 1/L^2$.

   (b) Terms of Higher-Power of φ : We may impose, on the acceptable 'topological' action of $\int d^2x\partial_a(\partial_a\varphi\cdot\varphi^n)$,the natural condition that the critical behaviour should not be influenced by the change of the regularization parameter: $L \rightarrow$ const $\times\ L$. This condition uniquely fix the power as $n = 1$. ($\int d^2x\partial_a(\partial_a\varphi_c\cdot{\varphi_c}^n) \sim\ L\cdot(1/L)(ln\ L/A)^n, L \rightarrow +\infty$.) Note that $\int d^2x\partial^2\varphi_c = -\int d^2x\sqrt{g}R|_{\varphi_c} = -8\pi$.

   Therefore no ambiguity exists, except the ξ-term, in the theory.

2. In the analysis of the ordinary conformal approach ,the kinetic term of φ in Liouville action is used only for the explanation of assigning the free-field form to the 2-point function of $\varphi(x)$: $<\varphi(x)\varphi(y)> \sim\ \ ln\ |x-y|$. The global(topological) effect,which is essential for the critical exponents such as the string susceptibility, is obtained not by the lagrangian but by the

requirement of the conformal symmetry (for the partition function). The present approach contrasts with this. We do not use the requirement of the conformal symmetry. Instead we directly use the lagrangian and its explicit Liouville solution which contains the essential part of the conformal symmetry. And the global aspect of the theory can be taken into account through the total derivative term in the lagrangian. Note that the infra-red regularization L in (16), besides the total derivative term itself, is important for the quantity $Z[A]$ to have the correct conformal behaviour for β (or w) $\rightarrow +0$. The present analysis manifestly reveals the importance of the infra-red regularization. This point was stressed by T.Yoneya[4].

3. We have examined only the classical configuration in the present paper. The quantum aspect is, of course, very important. One of us (S.I.) is preparing for the quantum analysis in the present formalism[18]. R^2-term suppresses the ultra-violet divergences quite well and makes the theory renormalizable. Besides the renormalizability, the unitarity problem is also important generally in the higher-derivative theories. In the present case of 2d R^2-gravity, it was argued in [4] , for the case of Loretzian metric, that the framework for the unitarity discussion (such as the meaning of state, wave functinal, etc.) should be first settled. This problem deserves further study.

4. We have chosen a constant curvature solution as the vacuum. The importance of the constant curvature configuration in 2d quantum gravity was stressed by A.H. Chamseddine[19] in the context of the conformal formalism of the ordinary (not R^2) 2d gravity. His model has an auxiliary scalar field ϕ : $\mathcal{L}_1 = \sqrt{g}\phi(R+\Lambda)$. Due to the presence of the auxiliary field, the model always has, at the classical level, a constant curvature configuration. He argues some difficulties in the conformal approach, such as the limitation on the target-space dimension in the string-terminology, are naturally resolved. R^2-gravity can be regarded as a kind of the above-mentioned model : $\mathcal{L}_2 = \sqrt{g}\{\phi(R+\Lambda)+c_1\phi^2\}$. Kawai-Nakayama[7] has taken this approach in their analysis. Further generalization of the above model, which includes more-higher derivative terms, has been studied in [20, 21, 22, 23]. These general models are interesting as future alternate theories when the 2d quantum gravity faces serious problems.

We have analysed Liouville theory induced by R^2-gravity, at the classical level. For the analysis we have presented the effective action formalism using $Y[A,\lambda]$ (9),which efficiently takes into account the area constraint. The features of three phases are explained theoretically. The importance of the total derivative term is stressed. The free parameter ξ is fixed to be 1 by comparing the present approximate result $Z[A]$ with the exact KPZ result at the 'classical' limit $c_m \rightarrow -\infty$. In particular the prediction about the expectation value of $< \int d^2x\sqrt{g}R^2 >$ well fits the data of the computer simulation for all real β-region. It makes sure of the validity of the semiclassical approach. The small discrepancy comes from the quantum effect.

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